B) DEFORMATION OF A CYLINDRICAL BODY IN THE PRESENCE OF GRAVITY

B.1) The model.

An elastic homogeneous isotropic body is considered (Fig.B1). Its initial shape is a right, vertical, **very thin** cylinder of radius equal to r and height equal to H. The base of the body is placed on the horizontal, absolutely rigid, plane x_1Ox_2. The deformation of the body due to its own weight follows to be studied and the final shape of the body into the final equilibrium state will be found. The approximations of the linear theory are assumed and the variation of the density is ignored. The problem is solved by following the next steps:

i) - the equations of equilibrium are used, the unknowns here being the components of the stress tensor σ ; these equations are processed according to the simplifying hypothesis of the problem;

ii) - by using the reversed HOOKE's law, the equations of equilibrium are processed in order to have only the components of the strain tensor ε as unknowns;

iii) - by using the definition of the strain tensor, the components of the displacement vector $\vec{u}$ are obtained and the final shape of the body is found.

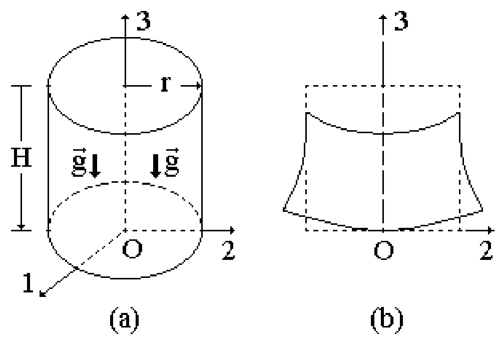

Fig.B.1. (a) A vertical cylinder lying on a rigid plane; (b) The final shape of a vertical cross section (solid line) with respect to the initial shape (dashed line). [NO SCALE]

B.2) The equations of equilibrium. Boundary conditions. Simplifying hypothesis.

A simplified approach can be derived by using cylindrical co-ordinates. However, the problem here is an introductory one. So these co-ordinates will be used later, in relation to other problems. The equations of equilibrium in Cartesian co-ordinates are

$$\begin{aligned} &\sigma_{11,1}+\sigma_{12,2}+\sigma_{13,3}=0 \\ &\sigma_{12,1}+\sigma_{22,2}+\sigma_{23,3}=0 \\ &\sigma_{13,1}+\sigma_{23,2}+\sigma_{33,3}-\rho g=0 \end{aligned} \tag{b1}$$

Here, δ is the density and g is the gravitational acceleration. The forces acting upon the body are the reaction force of the horizontal plane and the gravity of the cylinder. The boundary conditions are:

-on the lateral surface of the cylinder:

$$\sigma\,\vec{n}=\vec{0}\quad ,\text{for}\quad x_3\in[0,H],\quad x_1,x_2\in\Gamma \tag{b2}$$

-on the upper base of the cylinder

$$\sigma\,\vec{n}=\vec{0}\quad ,\text{for}\quad x_3=H,\quad x_1,x_2\in\Delta\qquad , \tag{b3}$$

Here, Δ is the disc of radius equal to r, having the centre at the origin of the co-ordinate system and the boundary denoted by Γ. The outer pointing normal at the lateral surface of the body is a linear combination with variable coefficients of the horizontal unit vectors, i.e.

$$\vec{n}=C_1(x_1,x_2)\vec{e}_1+C_2(x_1,x_2)\vec{e}_2 \tag{b4}$$

For $x_3 \in [0, H], \quad x_1, x_2 \in \Gamma$ eq.(b2) becomes

$$C_1(x_1,x_2)[\sigma_{11}\vec{e}_1 + \sigma_{12}\vec{e}_2 + \sigma_{13}\vec{e}_3] + C_2(x_1,x_2)[\sigma_{12}\vec{e}_1 + \sigma_{22}\vec{e}_2 + \sigma_{23}\vec{e}_3] = \vec{0} \qquad \text{(b5)}$$

The outer pointing normal at the upper base of the body is the unit vector $\vec{e}_3$. For $x_3 = H, \quad x_1, x_2 \in \Delta$, eq. (b3) gives

$$\sigma_{13}\vec{e}_1 + \sigma_{23}\vec{e}_2 + \sigma_{33}\vec{e}_3 = \vec{0} \qquad \text{(b6)}$$

Eq. (b5) is satisfied if the stress tensor has the form

$$\left[\sigma\right] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \qquad \text{(b7)}$$

on the lateral surface of the body.

Because the cylinder is a very thin one, the stress at its inner points is approximately the same one to the stress on the lateral surface. So, it is assumed that eq.(b7) holds inside the whole volume of the body. It follows eqs.(b1a)-(b1b) are identical verified. From eq. (b1c) it follows that

$$\frac{\partial \sigma_{33}}{\partial x_3} = \rho g, \quad \sigma_{33}(x_1, x_2, x_3 = H) = 0 \qquad \text{(b8)}$$

The problem represented by eq.(b8) has the next immediate solution

$$\sigma_{33}(x_1, x_2, x_3) = \rho g (x_3 - H) \qquad \text{(b9)}$$

The reversed HOOKE's law is

$$\begin{aligned}
\varepsilon_{11} &= \frac{1}{E}\left[(1+\nu)\sigma_{11} - \nu\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right)\right] \\
\varepsilon_{22} &= \frac{1}{E}\left[(1+\nu)\sigma_{22} - \nu\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right)\right] \\
\varepsilon_{33} &= \frac{1}{E}\left[(1+\nu)\sigma_{33} - \nu\left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right)\right] \\
\varepsilon_{12} &= \frac{1+\nu}{E}\sigma_{12}, \ \varepsilon_{23} = \frac{1+\nu}{E}\sigma_{23}, \ \varepsilon_{13} = \frac{1+\nu}{E}\sigma_{13}
\end{aligned} \qquad \text{(b10)}$$

Because

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad \text{(b11)}$$

eqs. (b10) lead to

$$\begin{aligned}
&\frac{\partial u_1}{\partial x_1} = \frac{\nu\rho g}{E}(H - x_3), \quad \frac{\partial u_2}{\partial x_2} = \frac{\nu\rho g}{E}(H - x_3), \quad \frac{\partial u_3}{\partial x_3} = \frac{\rho g}{E}(x_3 - H) \\
&\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 0, \quad \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = 0, \quad \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} = 0
\end{aligned} \qquad \text{(b12)}$$

By integrating eq.(b12) it follows that

$$u_1 = \frac{\nu\rho g}{E}(H - x_3)x_1 + f_1(x_2, x_3) \quad ,$$

$$u_2 = \frac{\nu\rho g}{E}(H - x_3)x_2 + f_2(x_1, x_3) \quad ,$$

$$u_3 = \frac{\rho g}{E}(\frac{x_3^2}{2} - Hx_3) + f_3(x_1, x_2) \quad , \tag{b13}$$

$$\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} = 0 \quad , \quad \frac{\partial f_2}{\partial x_3} + \frac{\partial f_3}{\partial x_2} = \frac{\nu\rho g}{E} x_2 \quad , \quad \frac{\partial f_3}{\partial x_1} + \frac{\partial f_1}{\partial x_3} = \frac{\nu\rho g}{E} x_1$$

Hence the displacement field is found if the unknown functions f_1, f_2, f_3 are finally obtained. Differentiating (b13e) with respect to x_1 and (b13f) with respect to x_2 and adding the results, it follows

$$2\frac{\partial^2 f_3}{\partial x_1 \partial x_2} + \frac{\partial}{\partial x_3}\left(\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x1}\right) = 0 \quad . \tag{b14}$$

So, using eq.(b13d) it follows

$$\frac{\partial^2 f_3}{\partial x_1 \partial x_2} = 0 \tag{b15}$$

From eq.(b15) it follows that

$$f_3(x_1, x_2) = h_1(x_1) + h_2(x_2) \qquad , \tag{b16}$$

where h_1, h_2 are two unknown functions, following to be found. Eqs.(b13e) and (b13f) give

$$\frac{\partial f_2(x_1, x_3)}{\partial x_3} = \frac{\nu\rho g}{E} x_2 - \frac{dh_2(x_2)}{dx_2} \quad , \quad \frac{\partial f_1(x_2, x_3)}{\partial x_3} = \frac{\nu\rho g}{E} x_1 - \frac{dh_1(x_1)}{dx_1} \tag{b17}$$

The left side of eq.(b17a) is represented by a function depending on x_1, x_3 only, while the right side is a function of x_2. Hence both sides are equal to a constant, i.e.

$$\frac{\partial f_2(x_1, x_3)}{\partial x_3} = -a_2 \quad , \quad \frac{dh_2}{dx_2} = \frac{\nu\rho g}{E} x_2 + a_2 \tag{b18}$$

It follows that

$$f_2(x_1, x_3) = -a_2 x_3 + g_2(x_1) \quad , h_2(x_2) = \frac{\nu\rho g}{2E} x_2^2 + a_2 x_2 + b_2 \tag{b19}$$

In a similar manner, eq.(b17b) gives

$$f_1(x_2, x_3) = -a_1 x_3 + g_1(x_2) \quad , h_1(x_1) = \frac{\nu\rho g}{2E} x_1^2 + a_1 x_1 + b_1 \tag{b20}$$

From eq. (b13d) it follows that

$$\frac{dg_1(x_2)}{dx_2} = -\frac{dg_2(x_1)}{dx_1} = K \tag{b21}$$

where K is a constant. Then

$$g_1(x_2) = Kx_2 + C_1 \quad , \quad g_2(x_1) = -Kx_1 + C_2 \tag{b22}$$

For simplicity, material co-ordinates are used to obtain the final expression of the displacement field

$$\vec{u}(\vec{X}) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \frac{\rho g}{E}\begin{pmatrix} \nu(H - X_3)X_1 \\ \nu(H - X_3)X_2 \\ \frac{X_3^2}{2} - HX_3 + \frac{\nu}{2}(X_1^2 + X_2^2) \end{pmatrix} + \begin{pmatrix} -a_1X_3 - KX_2 \\ -a_2X_3 + KX_1 \\ a_1X_1 + a_2X_2 \end{pmatrix} + \begin{pmatrix} C_2 \\ C_1 \\ b_1 + b_2 \end{pmatrix} \qquad \text{(b23)}$$

The first term in eq.(b23) is the true displacement, the last one is a translation while the second term is the rigid rotation

$$\begin{pmatrix} a_2 \\ -a_1 \\ K \end{pmatrix} \times \vec{X} \qquad \text{(b24)}$$

B.3) The final shape of the body.

a) The final shape of the upper base

Consider an arbitrary point of co-ordinates equal to $(X_1, X_2, X_3 = H)$. In the initial stage, it is placed on the upper base of the cylinder. Finally, the co-ordinates of the point are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \\ H \end{pmatrix} + \frac{\rho g}{E}\begin{pmatrix} 0 \\ 0 \\ \frac{\nu}{2}(X_1^2 + X_2^2) - \frac{H^2}{2} \end{pmatrix} \qquad \text{(b25)}$$

Hence

$$\begin{cases} x_1 = X_1 \\ x_2 = X_2 \\ x_3 = \frac{\nu\rho g}{2E}(X_1^2 + X_2^2) - \frac{\rho g}{2E}H^2 + H \end{cases} \qquad \text{(b26)}$$

From eq.(b26) it follows that

$$x_1^2 + x_2^2 = X_1^2 + X_2^2 = r^2 \quad . \qquad \text{(b27)}$$

Hence the circle representing the contour of the upper base remains a circle of the same radius. The plane of the circle is moving downward by a quantity equal to $\rho g(H^2 - \nu r^2)/(2E)$. The surface of the disc representing the upper base of the body is no longer a plane one. It becomes a rotational parabolic surface having the equation

$$x_3 = \frac{\nu\rho g}{2E}(x_1^2 + x_2^2) - \frac{\rho g}{2E}H^2 + H \qquad \text{(b28)}$$

b) The final shape of the lower base

Consider now an arbitrary point initially placed on the lower base of the body. The point has the co-ordinates equal to $(X_1, X_2, X_3 = 0)$. The final co-ordinate of the point are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \\ 0 \end{pmatrix} + \frac{\nu\rho g}{E}\begin{pmatrix} HX_1 \\ HX_2 \\ (X_1^2 + X_2^2)/2 \end{pmatrix} \qquad \text{(b29)}$$

Hence

$$\begin{cases} x_1 = (1+\nu\rho gH / E)X_1 \\ x_2 = (1+\nu\rho gH / E)X_2 \\ x_3 = \dfrac{\nu\rho g}{2E}(X_1^2 + X_2^2) \end{cases} \tag{b30}$$

From eq.(b30) it follows that

$$x_1^2 + x_2^2 = (1+\nu\rho gH / E)^2 (X_1^2 + X_2^2) = (1+\nu\rho gH / E)^2 r^2 \quad , \tag{31}$$

i.e. the circle representing the contour of the lower base remains a circle. The new radius is increased by a quantity equal to $\nu\rho gH / E$. The initial horizontal plane of the circle is uplifted by a quantity equal to $\nu\rho gr^2 / (2E)$. The surface of the disc representing the lower base becomes a rotational paraboloid having the equation

$$x_3 = \frac{\nu\rho g}{2E(1+\nu\rho gH / E)^2}(x_1^2 + x_2^2) \tag{b32}$$

c) The final shape of the lateral surface

Consider now a point initially placed on a generatrix line of the cylinder. Because of the cylindrical symmetry of the problem, the point having the initial co-ordinates equal to $(X_1 = 0, X_2 = r, X_3)$ is considered. Finally, that point has the position characterised by the co-ordinates

$$\begin{cases} x_1 = 0 \\ x_2 = r + \dfrac{\nu\rho g}{E}(H - X_3)r \\ x_3 = X_3 + \dfrac{\rho g}{E}\left(\dfrac{X_3^2}{2} - HX_3\right) + \dfrac{\nu\rho g}{2E} r^2 \end{cases} \tag{b33}$$

From (b33), it follows that the generatrix remains into the initial vertical plane. Its shape is changed from a straight line segment to a convex parabolic segment, having the equation

$$x_3 = \frac{E}{2\nu^2\rho g}\left(x_2 / r - 1\right)^2 - \frac{E}{\nu\rho g}\left(x_2 / r - 1\right) + H - \frac{\rho g}{2E}H^2 + \frac{\nu\rho g}{2E}r^2 \tag{b34}$$

OBSERVATION. On the lower base of the cylinder it is acting the reaction force of the rigid plane, equal to the weight of the body. When the surface of the base is decreasing, approaching the paraboloid of eq.(b32), the normal unit effort (equal to the weight divided by the contact area) is increasing. At a certain moment, its magnitude will exceed a yielding value of the material. Then, HOOKE’s law, valid in the elastic domain, will be no longer appropriate here.